OSCILLATORS IN RESONANCE

Antonio Elipe [1] and André Deprit[1,2]

[1]Grupo de Mecánica Espacial. Universidad de Zaragoza, 50009 Zaragoza. Spain

[2]National Institut of Standards and Technology, Washington D.C., U.S.A.

In this communication, we consider dynamical systems defined by Hamiltonians of the type

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1;$$

the principal part consists of a finite number of harmonic oscillators

$$\mathcal{H}_0 = \frac{1}{2} \sum_{1 \le j \le n} (X_j^2 + \omega_j^2 x_j^2) \tag{1}$$

while $\mathcal{H}_1(\mathbf{x}, \mathbf{X})$ is a weak perturbation depending on the coordinates $\mathbf{x} = (x_1, \ldots, x_n)$ and of the moments $\mathbf{X} = (X_1, \ldots, X_n)$.

A finite family $(1 \le j \le n)$ of canonical transformations

$$\begin{aligned} x_j &= [(2\,\Phi_j)/(\omega_j)]^{1/2} \sin\varphi_j, \\ X_j &= (2\,\omega_j\,\Phi_j)^{1/2} \cos\varphi_j \end{aligned} \tag{2}$$

due to Poincaré converts (1) into the function

$$\mathcal{H}_0 = \sum_{1 \le j \le n} \omega_j\,\Phi_j. \tag{3}$$

The perturbation is now a function of the angles $\varphi = (\varphi_1, \ldots, \varphi_n)$ and the actions $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_n)$. To a General Theory of Perturbation, Poincaré's variables create the usual difficulty: the perturbation may introduce zero divisors when the frequencies (ω_i) are commensurable.

For pairs of harmonic oscillators, except for the resonances 1:1 (Kummer [1]) where the difficulty has been solved exactly [4], authors follow the example of Contopoulos [2] and Verhulst [3] by assuming that the ration ω_1/ω_2

differs from the resonance by an infinitesimal detuning. This stratagem allows them to use the Birkhoff normalization or an equivalent averaging method near the resonance.

In this communication, we propose a change of variables that will handle the commensurabilities without recourse to a detuning.

It is based on the Lissajous variables introduced in [4] for a pair of oscillators of equal frequencies. The latter proved useful in Galactic Dynamics and more generally in Molecular Physics; they were even arranged in [5] to accommodate some situations in resonances (1:1:1) for three oscillators. But the resonance (1:1) is an exceptional event, and Molecular Physics is interested in a general theory involving any number of oscillators in any type of resonances. To give one example, we mention the famous Fermi resonance (2:1).

References

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